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A five-dimensional form of the Dirac equation

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Abstract. A Dirac equation in a covariant form with respect to proper orthochronous rotations in (4+1)-dimensional pseudo-orthogonal space, i.e. Minkowski space extended by one real dimension is introduced. It contains a five-vector potential with a non-electromagnetic fifth component. The invariance of this equation under the CPT transformation is conditioned by the assumption that the real fifth coordinate changes its sign under charge conjugation, and that it simultaneously changes its sign either under time reversal or under space inversion. The energy levels of an electron under the simultaneous action of Coulomb and central gravitational fields are determined. To this end, (1) new eigenspinors of the total angular momentum operator are derived, with non-zero entries in the first and fourth or in the second and third row of the column matrix and (2) a scalar function is constructed from doubly-periodic Jacobian elliptic functions which, in the limit of the vanishing modulus of the elliptic functions, replaces the function $\exp(\mathrm{i}\omega t)$ in the stationary-state solutions. The iterated five-dimensional equation contains the ten components of the antisymmetric field tensor. It also contains a term determining the potential energy operator of electron spin density in a non-electromagnetic field. The Pauli equation is derived from the five-dimensional equation, with the transformational characteristics of the original equation. It contains a spin–orbit coupling term depending on the non-electromagnetic potential.

1. Introduction

Dirac's paper [1] on the electron wave equation in de Sitter space contains an alternative form of his equation in Minkowski space, namely

$$[\gamma_\alpha(\partial_\alpha - \mathrm{i}a_\alpha) - \mathrm{i}\gamma_5\kappa]\Psi = 0 \tag{1}$$

with $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and $a_\alpha = (e/\hbar c)A_\alpha$, $\alpha = 1, \ldots, 4$, with e denoting the electron charge and A_α denoting the components of the four-potential, where x_1, x_2, x_3 are real and $x_4 = \mathrm{i}ct$, with c denoting the speed of light in the vacuum, and $\kappa = mc/\hbar$ with m denoting the electron rest mass. This equation appears by Dirac [1] for the free-electron case as the unnumbered equation at the bottom of p 663, and for an electron in an external field it follows from his equation (33) on p 664. In this paper a five-dimensional form of the Dirac equation is proposed on the basis of equation (1).

In the investigations of the electron wave equation preceding Dirac's paper [1], the fifth dimension was considered by a number of authors. Klein [2] introduced a five-dimensional wave equation in a covariant form, which furnished a link between five-dimensional general relativity in the Kaluza–Klein form [3] and quantum mechanics. His equation contains terms depending on the derivatives with respect to the fifth coordinate. These terms are neglected and, on the basis of certain assumptions, the present Klein–Gordon equation is deduced. The fifth dimension appears in the investigations of spinors in five-dimensional projective space and

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of the electron wave equation by Pauli and Solomon [4] and in a series of papers by Schouten and van Dantzig and Schouten from which we name [5–7] and by Pauli [8]. Pauli's aim was to demonstrate that the projective differential geometry with five homogeneous coordinates furnished a general method for a unified presentation of gravitation and electromagnetism on the level of classical field theory and that it can be applied in quantum theory. He considered a five-dimensional projective space of real coordinates, spanned by the five anticommuting matrices γ_1, γ_2, γ_3, γ_4, and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, which fulfil the defining condition of the sixteen-dimensional Clifford algebra. He considered the group of four-dimensional matrices S depending on the rotation angles in the five-dimensional space of homogeneous coordinates and the four-component spinors which under five-dimensional rotations transform according to those matrices S. Owing to the irreducibility of the S matrices, these spinors cannot be decomposed into a pair of two-component spinors. Such a decomposition takes place only when the subgroup of four-dimensional rotations is considered. The existence of the irrep S furnished for Pauli the departure point for the application of the homogeneous coordinates method to the derivation of Dirac's equation. The relation between a class of relativistic wave equations, of which the Dirac equation is the simplest, and the groups in five dimensions, in particular the $SO(4,1)$ group, was discussed by Lubański [9]. Relativistically invariant wave equations for particles with spin greater than $1/2$ were investigated by Bhabha [10, 11]. His basic idea was to introduce into this area the real Lie algebra $so(4,1)$. To every representation of $so(4,1)$ corresponds a Bhabha relativistic wave equation. The free-electron Dirac equation is a particular example of a Bhabha equation corresponding to the four-dimensional irreducible representation of the real Lie algebra $so(4,1)$. The investigations in Dirac's paper [1] were further developed by Gürsey and Lee [12]. Rotation groups in five and six dimensions were considered with reference to the Dirac equation by Fronsdal [13], by Barut [14] and by Bakri [15]. Bracken and Cohn [16, 17] studied the connection between the free-electron Dirac equation and the $SO(4,1)$ group. The free-electron Dirac equation was written in a five-dimensional form and the matrix transformations, belonging to the $SO(4,1)$ group, which preserve the Lorentz-invariant scalar product of the Dirac spinors were determined. A unified group-theoretical description of a certain type of canonical momentum-dependent transformations was given. A higher-dimensional form of Dirac equation was considered by Witten [18, 19] and that form is of interest in the developments of field theory [20].

In section 2 we consider equation (1) for the free-electron case and introduce into it the term $\gamma_5\partial/\partial x_5$ with real x_5. The requirement of invariance under the group of local gauge transformations in the five-dimensional pseudo-orthogonal space of the respective free-electron Lagrangian, leads to the Dirac equation depending on a five-vector potential, with a non-electromagnetic fifth component. This equation is covariant with respect to proper orthochronous rotations in $(4+1)$-dimensional space.

In section 3 we derive the iterated equation which contains the five-dimensional form of the Klein–Gordon equation. This equation also contains the components of the five-dimensional rotation of the potential five-vector, which determines the electromagnetic and non-electromagnetic fields. There appears a term which has the meaning of the potential energy operator of electron spin density in a non-electromagnetic field.

The conditions of invariance of the five-dimensional equation under the combined transformations of charge conjugation, space inversion and time reversal are examined in section 4. We find that the fifth coordinate has to change its sign under charge conjugation and that simultaneously it has to change its sign either under time reversal or under space inversion.

In section 5 we derive a new form of eigenspinors of the total angular momentum operator of an electron. These spinors are indispensable for the solution of the electron-in-a-central-field problem, when γ_5 appears in the Dirac equation.

The question arises about the role of the fifth coordinate in measurable quantities calculated on the basis of the five-dimensional equation. This is essential if we do not want to utilize the fifth coordinate only as a vehicle for introducing the fifth potential component and subsequently to assume independence of the potentials and of the spinors on it. We are dealing with an analogous situation in Nordström's theory [3, 21], where the fifth dimension serves for introducing the gravitational potential and after that the fifth coordinate is removed from equations by assuming that the derivatives with respect to it vanish. In the Kaluza–Klein type theories [3] the fifth dimension is compact. Since we are dealing with a flat space, that assumption is ruled out. We can, however, examine the consequences of the fifth coordinate being even under parity operation and odd under time reversal, passing from the coordinates $x_4 = \mathrm{i}ct$ and real x_5 to a complex variable $t+\mathrm{i}u$, with real t and u, and with u even under charge conjugation. The complex variable $t + \mathrm{i}u$ can be the argument of doubly-periodic Jacobian elliptic functions. This means that the function $\exp(\mathrm{i}\omega t)$ determining the time dependence of a stationary state in Minkowski space, will be replaced in (4 + 1)-dimensional pseudo-orthogonal space by a function constructed from Jacobian elliptic functions of the arguments $(\omega t \pm \mathrm{i}\omega' u)/2$, with real ω and ω'. For $u = 0$ and a vanishing modulus of the elliptic functions that function reduces to $\exp(\mathrm{i}\omega t)$. If the frequency ω' is much smaller than the frequency ω, i.e. in the limit of $\omega'/\omega \to 0$, which implies a vanishing modulus of the elliptic functions, the periodicity in the u-variable is removed. The influence of the fifth coordinate on the energy levels of an electron in Coulomb and central-gravitational field is proportional to the square of the modulus of the elliptic functions. Sections 6 and 7 are devoted to this problem.

In section 8 we derive the Pauli equation from the five-dimensional Dirac equation. The Pauli equation now contains terms connected with the non-electromagnetic fifth component of the five-potential.

Some of the results of this paper have been presented in [22].

2. Dirac equation in five dimensions

We consider equation (1) in the case of a free electron, i.e. for $a_\mu = 0$, $\mu = 1, \ldots, 4$, and include into it the term $\gamma_5 \partial/\partial x_5$, with a real coordinate x_5 obtaining the equation

$$(\gamma_\mu \partial_\mu - \mathrm{i}\gamma_5 \kappa)\Psi = 0 \tag{2}$$

where the index μ varies from one to five and the spinor Ψ depends on the components of the five-vector $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$, referred to the coordinate axes of (4 + 1)-dimensional pseudo-orthogonal space, i.e. Minkowski space extended by one real dimension. In the following we assume that a matrix representation is substituted for the γ_μ.

We now require the invariance under the local gauge transformations in the (4 + 1)-dimensional space, of the Lagrangian density $\mathcal{L}(\vec{x})$ connected with equation (2), i.e. of

$$\mathcal{L}(\vec{x}) = -\hbar c \Psi^\dagger(\vec{x})\gamma_4[\gamma_\mu \partial_\mu \Psi(\vec{x}) - \mathrm{i}\gamma_5 \kappa \Psi(\vec{x})] \tag{3}$$

where $\Psi^\dagger$ denotes the Hermitean-conjugate spinor.

It can be verified that the Euler–Lagrange equation for the adjoint spinor $\overline{\Psi} = \Psi^\dagger \gamma_4$ leads to equation (2). This Lagrangian is invariant under the global gauge transformation: $\Psi \to \Psi \exp(\mathrm{i}\beta)$ for any real, $\vec{x}$-independent parameter β. The invariance of the Lagrangian in equation (3) under the local gauge transformation

$$\Psi \to \Psi \exp(\mathrm{i}\beta(\vec{x})) \tag{4}$$

where $\beta(\vec{x})$ is an arbitrary, real, dimensionless, differentiable scalar function of the five-vector $\vec{x}$, is conditioned by the replacement of the derivatives ∂_μ in (3) by

$$\mathrm{D}_\mu = \partial_\mu - \frac{\mathrm{i}g_\mu}{\hbar c} A_\mu(\vec{x}) \qquad \mu = 1, \ldots, 5 \tag{5}$$

where $A_\mu(\vec{x})$, $\mu = 1, \ldots, 5$, form a 5-vector and g_μ are real constants, provided that $A_\mu(\vec{x})$ undergo a transformation of the form

$$A_\mu(\vec{x}) \to A'_\mu(\vec{x}) = A_\mu(\vec{x}) + \frac{\hbar c}{g_\mu}\partial_\mu\beta(\vec{x}). \tag{6}$$

The modified Lagrangian density $\mathcal{L}_{\text{mod}}(\vec{x})$ has the form

$$\mathcal{L}_{\text{mod}}(\vec{x}) = -\hbar c\Psi^\dagger(\vec{x})\gamma_4[\gamma_\mu\partial_\mu\Psi(\vec{x}) - \mathrm{i}\gamma_5\kappa\Psi(\vec{x})] + \mathrm{i}\Psi^\dagger(\vec{x})\gamma_4\gamma_\mu g_\mu A_\mu(\vec{x})\Psi(\vec{x}). \tag{7}$$

With $g_\mu = e$, the electron charge, for $\mu = 1, \ldots, 5$, we identify the 5-vector potential components:

$$(A_1, A_2, A_3, A_4, A_5) = (A_x, A_y, A_z, \mathrm{i}\varphi, m\chi/e) \tag{8}$$

with m denoting electron rest mass, where A_x, A_y, A_z are the Cartesian components of the electromagnetic vector potential, φ is the scalar electromagnetic potential and χ is a real non-electromagnetic scalar potential. With the modified Lagrangian in (7), from the Euler–Lagrange equation for the adjoint spinor $\overline{\Psi}$, we obtain the five-dimensional form of the Dirac equation in external fields

$$[\gamma_\mu(\partial_\mu - \mathrm{i}a_\mu) - \mathrm{i}\gamma_5\kappa]\Psi = 0 \tag{9}$$

where a_μ, $\mu = 1, \ldots, 4$ are those in (1) and $a_5 = m\chi/\hbar c$. The covariance of (9) under five-dimensional proper orthochronous rotations in $(4+1)$-dimensional space can be demonstrated without introducing a matrix representation for the γ_μ following the argument in [23]. We write: $a_\mu = \Omega_\mu$, $\mu = 1, \ldots, 4$ and $a_5 + \kappa = \Omega_5$, and then (9) takes the form

$$\gamma_\nu(\partial_\nu - \mathrm{i}\Omega_\nu)\Psi = 0. \tag{10}$$

In the rotated coordinate system we are dealing with the 'primed' quantities ∂'_μ and Ω'_μ which are 5-vectors, and with the spinor $\Psi'(\vec{x}') = T_{16}\Psi(\vec{x})$, where the transformation T_{16} depends on the elements $r_{\mu\nu}$ of the five-dimensional pseudo-orthogonal rotation matrix and on the basis elements of the Clifford algebra. Rewriting (10) in the rotated coordinate system and subsequently multiplying it from the left-hand side by the inverse T_{16}^{-1} of the transformation T_{16}, we obtain the equation

$$(T_{16}^{-1}\gamma_\mu\partial'_\mu T_{16} - \mathrm{i}T_{16}^{-1}\gamma_\mu\Omega'_\mu T_{16})\Psi(\vec{x}) = 0. \tag{11}$$

On the other hand, expressing ∂_ν and Ω_ν in (10) through the 'primed' quantities ∂'_μ and Ω'_μ we obtain from it the equation

$$(\gamma_\nu r_{\mu\nu}\partial'_\mu - \mathrm{i}\gamma_\nu r_{\mu\nu}\Omega'_\mu)\Psi(\vec{x}) = 0. \tag{12}$$

Comparing (11) and (12) we find the covariance condition

$$T_{16}^{-1}\gamma_\mu T_{16} = r_{\mu\nu}\gamma_\nu. \tag{13}$$

By the same token we have proved the covariance of (9) under rotations in the Minkowski subspace and in the subspace spanned by the coordinates x_1, x_2, x_3 and x_5. The Cayley–Klein parameters for proper, orthochronous rotations in $(4+1)$-dimensional pseudo-orthogonal space were determined in [24].

We determine the adjoint equation in the manner of [23], without introducing an irrep of the Clifford algebra. Writing

$$M = \gamma_\mu(\partial_\mu - \mathrm{i}a_\mu) - \mathrm{i}\gamma_5\kappa \tag{14}$$

for the operator in (9), we define the adjoint operator N which acts to the left with the help of the equality

$$\overline{u}(Mu) - (\overline{u}N)u = \partial_\mu S_\mu \tag{15}$$

for the divergence of the 5-vector $\vec{S}$, where $\overline{u}$ denotes the function adjoint with respect to the function u which has the form

$$u = c_0(\vec{x}) + \gamma_1 c_1(\vec{x}) + \cdots + \gamma_1\gamma_2\gamma_3\gamma_4 c_{15}(\vec{x}) \tag{16}$$

where $c_0(\vec{x}), \ldots, c_{15}(\vec{x})$ are complex functions of the 5-vector $\vec{x}$. To obtain the right-hand side of (15) we have to assume that the operator N of the adjoint equation is equal to

$$N = -(\partial_\mu + \mathrm{i}a_\mu)\gamma_\mu - \mathrm{i}\gamma_5\kappa \tag{17}$$

and, following [23], reverse in $\overline{u}$ the sequence of γs in all γ-products, together with the replacements

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4, \mathrm{i} \longrightarrow -\gamma_1, -\gamma_2, -\gamma_3, \gamma_4, -\mathrm{i} \tag{18}$$

which implies that in these products $\gamma_5 \to -\gamma_5$. When u and $\overline{u}$ are the solutions of the equation $Mu = 0$ and the adjoint equation

$$\overline{u}N = 0 \tag{19}$$

respectively, we obtain from (15) the continuity equation

$$\partial_\mu S_\mu = 0 \tag{20}$$

with the customary definition of the components of the 5-vector $\vec{S}$

$$S_\mu = \overline{u}\gamma_\mu u. \tag{21}$$

With a matrix irrep for the γs we have $\overline{u} \to \Psi^\dagger\gamma_4$.

Assuming that $a_\mu = 0$, $\mu = 1, 2, 3$, and that a_4 and a_5 are central fields we obtain from (9) the Hamiltonian in the form

$$H = \hbar c \sum_{k=1}^{3} \gamma_4\gamma_k\partial_k + V - \mathrm{i}\gamma_4\gamma_5(W + mc^2) \tag{22}$$

where $V(r) = -\mathrm{i}eA_4(r)$ and $W(r) = eA_5(r)$. For this case (9) can be rewritten in the form

$$H\Psi(\vec{r}, x_4, x_5) = -\hbar c(\partial_4 + \gamma_4\gamma_5\partial_5)\Psi(\vec{r}, x_4, x_5). \tag{23}$$

It can be verified that the total angular momentum operators $J_z = L_z + \hbar\sigma_z/2$ and $\vec{J}^2$ commute with the Hamiltonian operator in (22) as well as with the respective operator M in (14). The operator K

$$K = \hbar[(\vec{r} \times \mathrm{grad}) \cdot \vec{\gamma} - 1]\gamma_4 \tag{24}$$

with $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, (see [23, 25, 26]), does not commute with the Hamiltonian in (22), since γ_4 does not commute with $\gamma_4\gamma_5$.

3. The iterated equation

We follow the argument in [23] and act on (9) from the left side with the operator $\gamma_\nu \mathrm{D}_\nu - \mathrm{i}\gamma_5\kappa$, with $\mathrm{D}_\nu = (\partial_\nu - \mathrm{i}a_\nu)$, thus obtaining the equation

$$[\gamma_\nu\gamma_\mu \mathrm{D}_\nu \mathrm{D}_\mu - 2\mathrm{i}\kappa(\partial_5 - \mathrm{i}a_5) - \kappa^2]\Psi = 0. \tag{25}$$

The first term in the square brackets of this equation for $\mu = \nu$, together with the term $-\kappa^2$, yields the five-dimensional form of the Klein–Gordon equation

$$\left[\sum_{\mu=1}^{5} \mathrm{D}_\mu^2 - \kappa^2\right]\Psi = 0. \tag{26}$$

For $\mu \neq \nu$ we obtain from the first term in the square brackets of (25) the expressions

$$-\mathrm{i}\gamma_\mu\gamma_\nu(\partial_\mu a_\nu - \partial_\nu a_\mu)\Psi. \qquad (27)$$

The term in the brackets in (27) is a component of the five-dimensional rotation of the 5-vector $\vec{A} = (A_1, A_2, A_3, A_4, A_5)$ in (8), multiplied by $e/\hbar c$. In a Cartesian coordinate system, with $(A_1, A_2, A_3) = (A_x, A_y, A_z)$ we obtain the magnetic induction components:

$$\partial_\mu A_\nu - \partial_\nu A_\mu = B_\rho \qquad \mu, \nu, \rho = 1, 2, 3 \qquad (28)$$

and the electric field components:

$$\partial_\mu A_4 - \partial_4 A_\mu = -\mathrm{i}E_\mu \qquad \mu = 1, 2, 3. \qquad (29)$$

We further have

$$\partial_\mu A_5 - \partial_5 A_\mu = -G_\mu \qquad \mu = 1, 2, 3 \qquad (30)$$

and for $\mu = 4$ and $\nu = 5$

$$\partial_4 A_5 - \partial_5 A_4 = -\mathrm{i}G_0 \qquad (31)$$

where G_μ and G_0 are real components of new fields connected with electromagnetic and non-electromagnetic potential. The signs on the right-hand sides of (30) and (31) are written by analogy with (29), however, they have to be the same if Maxwell–Nordström equations [3, 21] for the electromagnetic and gravitational field are to follow from (28) through (31). The second-rank antisymmetric tensor given in (28) through (31) is covariant with respect to proper, orthochronous rotations in (4+1)-dimensional pseudo-orthogonal space. The behaviour of the tensor components in (30) and (31) under space inversion and time reversal is determined in section 4. The term $2\mathrm{i}\kappa\partial_5$ in the square brackets in (25) will be examined in section 6. The term $-2\kappa a_5$ is another term connected with the fifth dimension.

We now consider the operator in (27) for $\mu = 1, 2, 3, 4$ and $\nu = 5$. Dividing this operator by $2m/\hbar^2$ we obtain the operator

$$\frac{\mathrm{i}\hbar^2}{2m}\sum_{\mu=1}^{4}\gamma_5\gamma_\mu\left(\frac{m}{\hbar c}\partial_\mu\chi - \partial_5 a_\mu\right) \qquad (32)$$

which has the dimension of energy. When a_μ, $\mu = 1, 2, 3, 4$ are independent of x_5 and χ is independent of x_4 we obtain from (32) the operator of the form

$$\frac{\hbar}{2c}\gamma_4\vec{\sigma}\cdot\mathrm{grad}\,\chi(\vec{r}) \qquad (33)$$

since $\mathrm{i}\gamma_5\gamma_1 = -\mathrm{i}\gamma_4\gamma_2\gamma_3 = \gamma_4\sigma_x$, etc. In (33) we are dealing with the potential energy operator of the electron spin density in a non-electromagnetic scalar field. This term does not appear in the Pauli equation which will be derived in section 8. The reason for this is that the Pauli equation is connected with the quaternion group, while the iterated Dirac equation in five dimensions, owing to the presence of γ_5 in equation (32) requires the sedenion group. In the Minkowski space, the iterated Dirac equation is connected with the biquaternion group [23].

4. The invariance under CPT transformation

The basic assumption is that the fifth dimension can be affected by the operations of charge conjugation C, space inversion P and time reversal T. Consequently, these operations, apart from their customary action in four-dimensional spacetime may also influence x_5 and a_5 in the five-dimensional form of the Dirac equation in (9). We will examine on which assumptions concerning the behaviour of x_5 and a_5, the invariance of that equation under the CPT operation, extended on the fifth dimension, can be achieved. If that invariance could not be obtained with any assumptions about the behaviour of x_5 and a_5 under the operations C, P and T then, as it seems, the five-dimensional equation in (9) would not necessarily be disproved. If, however, that invariance takes place, provided that x_5 and a_5 behave under the operations C, P and T in a definite way, this may be interpreted as a hint that the invariance requirement is justified. The respective properties of x_5 can be essential in an attempt at interpreting the role of the fifth dimension in the Dirac equation in (9).

In the demonstration of the invariance of (9) under the extended CPT transformation we shall utilize the irrep for the γ_αs in (59) of section 5. Our mode of reasoning will be analogous to that in [27, 29].

The postulated charge-conjugate equation with the respect to (9) has the form

$$[\gamma_\alpha(\partial_\alpha + \mathrm{i}a_\alpha) - \gamma_5(\partial_5 + \mathrm{i}a_5) - \mathrm{i}\gamma_5\kappa]\Psi_c = 0 \tag{34}$$

where α varies from one to four, with $\Psi_c = C\Psi$, where C denotes the charge conjugation operation. In (34) we have assumed that x_5 is odd and a_5 is even under charge conjugation

$$Cx_5 = -x_5C \qquad Ca_5 = a_5C. \tag{35}$$

Equations (35) are the necessary condition for the required equivalence of the charge-conjugate equation with the adjoint equation in (19). We consider the adjoint equation (19) in the matrix form, with the γ_αs in (59) and with $\overline{u} = \Psi^\dagger\gamma_4$. We multiply this equation by (-1), transpose it and subsequently multiply the transposed equation from the left side by a still undetermined product of γ_αs denoted by F, thus obtaining the equation

$$[(\partial_\mu + \mathrm{i}a_\mu)F\tilde{\gamma}_\mu\tilde{\gamma}_4 + \mathrm{i}F\tilde{\gamma}_5\tilde{\gamma}_4\kappa]\Psi^* = 0 \tag{36}$$

where μ varies from one to five, and $\tilde{\gamma}_\mu$ denotes a transposed matrix and Ψ^* denotes the complex conjugate spinor. The charge-conjugate equation (34) turns into (36) if we have

$$F\tilde{\gamma}_\alpha\tilde{\gamma}_4\Psi^* = \gamma_\alpha\Psi_c \qquad \alpha = 1, \ldots, 4 \tag{37}$$

and

$$F\tilde{\gamma}_5\tilde{\gamma}_4\Psi^* = -\gamma_5\Psi_c \tag{38}$$

or

$$\Psi_c = -\gamma_5F\tilde{\gamma}_5\tilde{\gamma}_4\Psi^*. \tag{39}$$

From (37) and (38) we obtain the condition for F

$$\gamma_5F\tilde{\gamma}_\alpha = \tilde{\gamma}_\alpha F\gamma_5 \qquad \alpha = 1, \ldots, 4. \tag{40}$$

We further require that

$$\Psi_c^\dagger\Psi_c = \Psi^\dagger\Psi = (\Psi^\dagger\Psi)^* \tag{41}$$

and from (39) and (41) find the second condition for F

$$\gamma_4\gamma_5F^\dagger F\gamma_5\gamma_4 = 1 \tag{42}$$

where $F^\dagger$ denotes the Hermitean conjugate of F, from which it follows that

$$F^\dagger F = 1. \tag{43}$$

From (40) and (43) we finally find that

$$F = \gamma_2\gamma_3 \tag{44}$$

which fulfils (43). From (39) and (44) we obtain the expression for the charge-conjugate spinor

$$\Psi_c = \gamma_1\gamma_5\Psi^* = \gamma_1\gamma_5K\Psi = C\Psi \tag{45}$$

where K denotes the conjugate-complex operation.

If we have assumed that $Cx_5 = x_5C$ and $Ca_5 = -a_5C$, which would require an a_5 depending on the electron charge, we would have obtained $F = 0$, which excludes the equivalence of (19) and (34).

At this point the argument bifurcates depending on whether the space inversion is delimited to the coordinates x_1, x_2, x_3 or extended on x_5. We firstly consider the case when space inversion does not affect x_5. The customary parity operation applied to $\vec{r}$ and to a_α, $\alpha = 1, \ldots, 4$, is supplemented with the respective rules:

$$Px_5 = x_5P \qquad Pa_5 = a_5P. \tag{46}$$

Consequently, under the parity operation P, (9) turns into the equation

$$\left[-\sum_{k=1}^{3}\gamma_k(\partial_k - \mathrm{i}a_k) + \sum_{m=4}^{5}\gamma_m(\partial_m - \mathrm{i}a_m) - \mathrm{i}\gamma_5\kappa\right]P\Psi = 0. \tag{47}$$

Writing $P\Psi = D\Psi$, with D denoting an undetermined product of γ_μs which anticommutes with γ_k, $k = 1, 2, 3$, and at the same time commutes with γ_4 and γ_5, which is required for transforming (47) to its original form (9), we can verify that such a D does not exist. This means that the five-dimensional form of the Dirac equation in (9) is not invariant under the space-inversion operation. However, we will accept as the definition of action of the parity operation on the spinor Ψ the equality

$$P\Psi(\vec{r}, x_4, x_5) = \gamma_4\Psi(-\vec{r}, x_4, x_5) \tag{48}$$

which, without x_5, implies the invariance under the space-inversion operation of the original form of the Dirac equation (58). Equation (48) will be assumed in the following.

We now consider the charge-conjugate equation (34) with the spinor in (45) and utilizing (48) act on it with the parity operation P thus obtaining the equation

$$[\gamma_\mu(\partial_\mu + \mathrm{i}a_\mu) + \mathrm{i}\gamma_5\kappa]\Psi_c = 0 \tag{49}$$

where μ varies from one to five. We now assume that x_5 is odd and a_5 is even under the time reversal operation

$$Tx_5 = -x_5T \qquad Ta_5 = a_5T. \tag{50}$$

Under the time-reversal operation (49) then takes the form

$$\left[\sum_{k=1}^{3}\gamma_k(\partial_k - \mathrm{i}a_k) - \sum_{m=4}^{5}\gamma_m(\partial_m - \mathrm{i}a_m) + \mathrm{i}\gamma_5\kappa\right]T\Psi_c = 0. \tag{51}$$

Equation (51) turns into (9) if $T\Psi_c = G\Psi$, with $G\gamma_k = \gamma_kG$, $k = 1, 2, 3$, and $G\gamma_m = -\gamma_mG$, $m = 4, 5$, which implies that $G = \gamma_1\gamma_2\gamma_3$. Consequently, the operation CPT does not alter (9). From $T\Psi_c = \gamma_1\gamma_2\gamma_3\Psi$ and Ψ_c given in (45) we obtain $T = \gamma_4\gamma_1K$ with $T^2 = -1$. It can be verified that when equations (46) are assumed the five-dimensional form of the Dirac equation is invariant under the CPT transformation only on the assumption that the fifth coordinate is odd under time reversal and under charge conjugation.

When space inversion operation affects x_5, we replace (46) by the respective conditions:

$$Px_5 = -x_5P \qquad Pa_5 = a_5P \tag{52}$$

and then under the (four-dimensional) parity operation (9) takes the form

$$\left[-\sum_{k=1}^{3}\gamma_k(\partial_k - \mathrm{i}a_k) + \gamma_4(\partial_4 - \mathrm{i}a_4) + \gamma_5(-\partial_5 - \mathrm{i}a_5) - \mathrm{i}\gamma_5\kappa\right]P\Psi = 0 \tag{53}$$

and, as it was with (47), it is not invariant. However, by analogy with (48) we accept as the definition of action of the parity operation on the spinor Ψ the equality

$$P\Psi(\vec{r}, x_4, x_5) = \gamma_4\Psi(-\vec{r}, x_4, -x_5). \tag{54}$$

We now consider the charge-conjugate (34) with the spinor in (45) and act on it with the customary parity operation supplemented with (52) and (54), thus obtaining the equation

$$\left[\sum_{\alpha=1}^{4}\gamma_\alpha(\partial_\alpha + \mathrm{i}a_\alpha) - \gamma_5(\partial_5 - \mathrm{i}a_5) + \mathrm{i}\gamma_5\kappa\right]\Psi_c = 0. \tag{55}$$

If we now assume that both x_5 and a_5 are even under the time reversal operation

$$Tx_5 = x_5T \qquad Ta_5 = a_5T \tag{56}$$

and apply that operation to (55), we again obtain (51) which turns into (9). Consequently, the operation CPT with the action of P extended in (52), and with (56) supplementing the customary action of the time reversal operation, does not alter (9).

It is seen that the invariance of the five-dimensional form of the Dirac equation can be achieved on two assumptions: either with x_5 even under P-operation and odd under T-operation, i.e. with the conditions in (46) and (50), or with x_5 odd under P-operation and even under T-operation, i.e. with the conditions in (52) and (56). In both cases the odd character of x_5 under charge conjugation, i.e. the conditions in (35) are indispensable.

We now can determine the behaviour of the field tensor components in (30) and (31) of section 3. If x_5 is even under space inversion and odd under time reversal ((46) and (50)), then under P- or T-operation the left-hand sides of (30) and (31) have a definite character: they either change their sign or not. If x_5 is odd under P-operation and even under T-operation ((52) and (56)), the left-hand sides of (30) and (31) do not have a definite character, since under each of these operations, in both equations one term changes its sign while the other does not. This property of the field tensor seems to speak in favour of x_5 being even under P-operation and odd under T-operation. These two properties of the fifth coordinate will represent the starting point of section 6.

5. The eigenspinors of the total angular momentum operator

For an electron in the Coulomb field the integration of the alternative form of the Dirac equation in (1) or of (9), even without the term $(\partial_5 - \mathrm{i}a_5)$, cannot be accomplished with the help of the eigenspinors constructed from the two-component eigenspinors of the Pauli equation. This is seen when a standard representation for the γ_αs is introduced into (1). To this end we will utilize the matrices assigned to γ_1, γ_2, γ_3 and γ_4 on the p 369 of [27] or p 121 of [28], or the matrices on the p 58 of [25] after the multiplication by $(-\mathrm{i})$. With $a_\mu = 0$, $\mu = 1, 2, 3$ and $a_4 = \mathrm{i}V/\hbar c$, with the x_4-dependence of the spinor in the form $\exp(-Ex_4/\hbar c)$, in the Cartesian coordinate system, we then obtain from (1) the set of equations:

$$\begin{aligned}
&(\partial_x - \mathrm{i}\partial_y)\phi_4 + \partial_z\phi_3 - \mathrm{i}\frac{E-V}{\hbar c}\phi_1 - \kappa\phi_3 = 0\\
&(\partial_x + \mathrm{i}\partial_y)\phi_3 - \partial_z\phi_4 - \mathrm{i}\frac{E-V}{\hbar c}\phi_2 - \kappa\phi_4 = 0\\
&-(\partial_x - \mathrm{i}\partial_y)\phi_2 - \partial_z\phi_1 + \mathrm{i}\frac{E-V}{\hbar c}\phi_3 - \kappa\phi_1 = 0\\
&-(\partial_x + \mathrm{i}\partial_y)\phi_1 + \partial_z\phi_2 + \mathrm{i}\frac{E-V}{\hbar c}\phi_4 - \kappa\phi_2 = 0
\end{aligned} \tag{57}$$

with ϕ_α, $\alpha = 1, \ldots, 4$ denoting the space-dependent part of the spinor components, which differs in the indices of the spinor components connected with the mass term κ, from the set of equations which would have been obtained from the original Dirac equation

$$(\gamma_\alpha \mathrm{D}_\alpha + \kappa)\Psi = 0 \tag{58}$$

where $\alpha = 1, \ldots, 4$, with the same irrep for γ_1, γ_2, γ_3 and γ_4. In contradistinction to the case of the respective set of equations following from (58), in each of the four equations (57) the spinor components connected with ∂_z and with κ are the same. Consequently, the spherical harmonic functions cannot be separated out from these equations and the couple of equations containing only the two radial functions cannot be obtained.

The integration of the set of equations following from (1) for an electron in the Coulomb field is conditioned by the exchange of the matrices assigned to γ_3 and γ_4 in the above quoted representations [25, 27, 28]. This can be accomplished by an appropriate unitary transformation. Consequently, we obtain the following representation for the generators of the Clifford group C_4:

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix} \qquad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$
$$\gamma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \gamma_4 = \begin{bmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{bmatrix} \tag{59}$$

We observe that in this irrep the Hamiltonian in (22) is Hermitean. From (59) we can calculate the respective matrices for the spin operator $\vec{\sigma}$ components: $\sigma_x = -\mathrm{i}\gamma_2\gamma_3$, $\sigma_y = -\mathrm{i}\gamma_3\gamma_1$ and $\sigma_z = -\mathrm{i}\gamma_1\gamma_2$, and with $\vec{L}$ denoting the angular momentum operator we find that the matrix for $\vec{L}\cdot\vec{\sigma}$ has the form

$$\vec{L}\cdot\vec{\sigma} = \begin{bmatrix} L_z & 0 & 0 & -\mathrm{i}L_- \\ 0 & -L_z & \mathrm{i}L_+ & 0 \\ 0 & -\mathrm{i}L_- & L_z & 0 \\ \mathrm{i}L_+ & 0 & 0 & -L_z \end{bmatrix} \tag{60}$$

where $L_\pm = L_x \pm \mathrm{i}L_y$.

The eigenspinors of the operator $J_z = L_z + S_z$, with $\vec{S} = \hbar\vec{\sigma}/2$, which are of the form

$$\Phi = \begin{bmatrix} C_1 \exp[\mathrm{i}(m_j - \frac{1}{2})\varphi] \\ C_2 \exp[\mathrm{i}(m_j + \frac{1}{2})\varphi] \\ C_3 \exp[\mathrm{i}(m_j - \frac{1}{2})\varphi] \\ C_4 \exp[\mathrm{i}(m_j + \frac{1}{2})\varphi] \end{bmatrix} \tag{61}$$

with $C_p = C_p(r, \vartheta)$, $p = 1, 2, 3, 4$, and φ,ϑ denoting the angles of spherical coordinates, and the half-integral $m_j = m_l \pm \frac{1}{2}$, can be rewritten in the form

$$\Phi = \begin{bmatrix} f_j(r) Y_l^{m_j-1/2}(\vartheta,\varphi) \\ g_j(r) Y_{l'}^{m_j+1/2}(\vartheta,\varphi) \\ f_{1j}(r) Y_{l'}^{m_j-1/2}(\vartheta,\varphi) \\ g_{1j}(r) Y_l^{m_j+1/2}(\vartheta,\varphi) \end{bmatrix} \tag{62}$$

where $f_j(r)$,$g_j(r)$,$f_{1j}(r)$ and $g_{1j}(r)$ are the radial functions while l and l' are the orbital quantum numbers. The indices l and l' are assigned in agreement with the form of the operator

$\vec{L}\cdot\vec{\sigma}$ in (60). For brevity we will omit in the following formulae the index j by the functions f_j, f_{1j} and g_j, g_{1j}. Operating with $\vec{J}^2 = (\vec{L}^2 + 3\hbar^2/4 + \hbar\vec{L}\cdot\vec{\sigma})$ on the spinor (62) and remembering that $\vec{J}^2\Phi = \hbar^2 j(j+1)\Phi$, we obtain the equations:

$$\begin{aligned} &Af(r) + \mathrm{i}Cg_1(r) = j(j+1)f(r)\\ &B'g(r) - \mathrm{i}D'f_1(r) = j(j+1)g(r)\\ &A'f_1(r) + \mathrm{i}C'g(r) = j(j+1)f_1(r)\\ &Bg_1(r) - \mathrm{i}Df(r) = j(j+1)g_1(r) \end{aligned} \tag{63}$$

with:

$$\begin{aligned} A &= l(l+1) + \tfrac{3}{4} + (m_j - \tfrac{1}{2})\\ B &= l(l+1) + \tfrac{3}{4} - (m_j + \tfrac{1}{2})\\ C &= D = \sqrt{(l + m_j + \tfrac{1}{2})(l - m_j + \tfrac{1}{2})} \end{aligned} \tag{64}$$

and with A', B', C' and D' in (63) obtained from A, B, C, and D, respectively, by replacing l by l'. The condition for the non-zero solutions of (63) has the form

$$\{[A - j(j+1)][B - j(j+1)] - CD\}\{[A' - j(j+1)][B' - j(j+1)] - C'D'\} = 0 \tag{65}$$

The solution of equations (63) for $l = j - \frac{1}{2}$ and $l' = j + \frac{1}{2}$ leads to the spinors:

$$\Phi_1^{(a)} = \begin{bmatrix} f_j(r)Y_{j-\frac{1}{2}}^{m_j-\frac{1}{2}}\\ 0\\ 0\\ -\mathrm{i}\sqrt{\dfrac{j-m_j}{j+m_j}}\,f_j(r)Y_{j-\frac{1}{2}}^{m_j+\frac{1}{2}} \end{bmatrix} \qquad \Phi_2^{(a)} = \begin{bmatrix} 0\\ g_j(r)Y_{j+\frac{1}{2}}^{m_j+\frac{1}{2}}\\ -\mathrm{i}\sqrt{\dfrac{j-m_j+1}{j+m_j+1}}\,g_j(r)Y_{j+\frac{1}{2}}^{m_j-\frac{1}{2}}\\ 0 \end{bmatrix} \tag{66}$$

where we have restored the index j by the functions f and g. A linear combination of spinors from (66) also is a solution of the set of (63).

For $l = j + \frac{1}{2}$, $l' = j - \frac{1}{2}$, the solution of (63) yields the spinors:

$$\Phi_1^{(b)} = \begin{bmatrix} f_j(r)Y_{j+\frac{1}{2}}^{m_j-\frac{1}{2}}\\ 0\\ 0\\ \mathrm{i}\sqrt{\dfrac{j+m_j+1}{j-m_j+1}}\,f_j(r)Y_{j+\frac{1}{2}}^{m_j+\frac{1}{2}} \end{bmatrix} \qquad \Phi_2^{(b)} = \begin{bmatrix} 0\\ g_j(r)Y_{j-\frac{1}{2}}^{m_j+\frac{1}{2}}\\ \mathrm{i}\sqrt{\dfrac{j+m_j}{j-m_j}}\,g_j(r)Y_{j-\frac{1}{2}}^{m_j-\frac{1}{2}}\\ 0 \end{bmatrix} \tag{67}$$

A linear combination of the spinors (67) also is a solution of the set of (63). The solution of (63) for $l = l' = j - \frac{1}{2}$ is given by the pair $\Phi_1^{(a)}$ and $\Phi_2^{(b)}$ and by its linear combination and for $l = l' = j + \frac{1}{2}$ by the pair $\Phi_1^{(b)}$ and $\Phi_2^{(a)}$ and by its linear combination.

For a free electron with the irrep in (59) we find for positive energies and positive or negative helicity the respective spinor amplitudes C^+ and C^-:

$$C^+ = R^{-1}\begin{bmatrix} \eta\cos\frac{1}{2}\vartheta\ \mathrm{e}^{-\mathrm{i}\varphi/2}\\ -\mathrm{i}\sin\frac{1}{2}\vartheta\ \mathrm{e}^{\mathrm{i}\varphi/2}\\ -\cos\frac{1}{2}\vartheta\ \mathrm{e}^{-\mathrm{i}\varphi/2}\\ \mathrm{i}\eta\sin\frac{1}{2}\vartheta\ \mathrm{e}^{\mathrm{i}\varphi/2} \end{bmatrix} \qquad C^- = R^{-1}\begin{bmatrix} \eta\sin\frac{1}{2}\vartheta\ \mathrm{e}^{-\mathrm{i}\varphi/2}\\ -\mathrm{i}\cos\frac{1}{2}\vartheta\ \mathrm{e}^{\mathrm{i}\varphi/2}\\ \sin\frac{1}{2}\vartheta\ \mathrm{e}^{-\mathrm{i}\varphi/2}\\ -\mathrm{i}\eta\cos\frac{1}{2}\vartheta\ \mathrm{e}^{\mathrm{i}\varphi/2} \end{bmatrix} \tag{68}$$

where $R = \sqrt{V_0(1+\eta^2)}$ and V_0 is the normalization volume, while $v/c = 2\eta/(1+\eta^2)$, where v denotes the absolute value of electron velocity and c is the speed of light. In the non-relativistic case, when $v \ll c$, the respective large components of the two spinor amplitudes are:

$$C^+ = R^{-1}\begin{bmatrix} 0 \\ -\mathrm{i}\sin\frac{1}{2}\vartheta\,\mathrm{e}^{\mathrm{i}\varphi/2} \\ -\cos\frac{1}{2}\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi/2} \\ 0 \end{bmatrix} \qquad C^- = R^{-1}\begin{bmatrix} 0 \\ -\mathrm{i}\cos\frac{1}{2}\vartheta\,\mathrm{e}^{\mathrm{i}\varphi/2} \\ \sin\frac{1}{2}\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi/2} \\ 0 \end{bmatrix} \tag{69}$$

having the form of the spinors $\Phi_2^{(a)}$ and $\Phi_2^{(b)}$ in (66) and (67), respectively. For negative energies the respective spinor amplitudes D^+ and D^- are:

$$D^+ = R^{-1}\begin{bmatrix} \cos\frac{1}{2}\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi/2} \\ \mathrm{i}\eta\sin\frac{1}{2}\vartheta\,\mathrm{e}^{\mathrm{i}\varphi/2} \\ \eta\cos\frac{1}{2}\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi/2} \\ \mathrm{i}\sin\frac{1}{2}\vartheta\,\mathrm{e}^{\mathrm{i}\varphi/2} \end{bmatrix} \qquad D^- = R^{-1}\begin{bmatrix} -\sin\frac{1}{2}\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi/2} \\ -\mathrm{i}\eta\cos\frac{1}{2}\vartheta\,\mathrm{e}^{\mathrm{i}\varphi/2} \\ \eta\sin\frac{1}{2}\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi/2} \\ \mathrm{i}\cos\frac{1}{2}\vartheta\,\mathrm{e}^{\mathrm{i}\varphi/2} \end{bmatrix} \tag{70}$$

Their large components have the form of $\Phi_1^{(a)}$ and $\Phi_1^{(b)}$ in (66) and (67), respectively.

There arises the question about the transformation of these four-component spinors under three-dimensional spatial rotations. This can be answered by reference to [24] where the group of rotations in Minkowski space extended by one real dimension, i.e. in $(4+1)$-dimensional pseudo-orthogonal space was discussed. There is a four-dimensional irreducible double-valued representation of these rotations, according to which first-rank, four-dimensional spinors transform. For the subgroup of three-dimensional spatial rotations, this four-dimensional irrep reduces to a block diagonal matrix, with two $SU(2)$ matrices along the diagonal. Under three-dimensional rotations, the four-dimensional spinors transform according to that block-diagonal matrix.

6. The variable $(\omega t + \mathrm{i}\omega' u)/2$

Referring to the end of section 4 we accept that the coordinate x_5 is time-reversal and charge-conjugation odd and space-inversion even. We write $x_5 = qcu$, with c denoting the speed of light in the vacuum, with $Cq = -qC$ and $q^2 = 1$ and $Cu = uC$, with u expressed in seconds. Since the variables t and u behave in the same way under the operations C, P and T, we can introduce the complex variables:

$$\xi = \tfrac{1}{2}(\omega t + \mathrm{i}\omega' u) \qquad \xi^* = \tfrac{1}{2}(\omega t - \mathrm{i}\omega' u) \tag{71}$$

with real frequencies ω and ω'. We then have:

$$\partial_t = \tfrac{1}{2}\omega(\partial_\xi + \partial_{\xi^*}) \qquad \partial_u = \mathrm{i}\tfrac{1}{2}\omega'(\partial_\xi - \partial_{\xi^*}) \tag{72}$$

and with $x_4 = \mathrm{i}ct$, $x_5 = qcu$ we obtain

$$\partial_4 \pm \mathrm{i}\partial_5 = \frac{1}{2\mathrm{i}c}\left[\left(\omega \mp \frac{\mathrm{i}}{q}\omega'\right)\partial_\xi + \left(\omega \pm \frac{\mathrm{i}}{q}\omega'\right)\partial_{\xi^*}\right]. \tag{73}$$

These derivatives appear in the matrix form of the five-dimensional Dirac equation. We now consider the scalar function F of the complex variables ξ and ξ^* in (71)

$$F(\xi, \xi^*) = (\mathrm{cn}\,\xi + \mathrm{i}\,\mathrm{sn}\,\xi)(\mathrm{cn}\,\xi^* + \mathrm{i}\,\mathrm{sn}\,\xi^*) \tag{74}$$

where $\mathrm{cn}\,\xi$ and $\mathrm{sn}\,\xi$ are Jacobian elliptic functions, referred to the orthogonal axes t and $\mathrm{i}u$, with real periods along the t-axis and imaginary periods along the $\mathrm{i}u$-axis. The function $F(\xi, \xi^*)$

does not vanish in the whole complex plane. From (74) and from the equality: $\mathrm{sn}^2\,\xi + \mathrm{cn}^2\,\xi = 1$, it follows that for all values of ξ and ξ^* we have

$$|F(\xi,\xi^*)|^2 = 1 \qquad \text{for all } \xi, \xi^*. \tag{75}$$

For $u = 0$, and for the vanishing modulus k of the elliptic functions we have

$$F(\xi,\xi^*) \to \exp(\mathrm{i}\omega t) \qquad u = 0,\ k \to 0. \tag{76}$$

Recalling that we have: $d_\xi \mathrm{sn}\,\xi = \mathrm{cn}\,\xi\,\mathrm{dn}\,\xi$, $d_\xi \mathrm{cn}\,\xi = -\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi$ where $\mathrm{dn}^2\,\xi = 1 - k^2\mathrm{sn}^2\,\xi$, we obtain from (73) and (74) the expression

$$(\partial_4 \pm \mathrm{i}\partial_5)F(\xi,\xi^*) = \frac{1}{2c}\left[\omega(\mathrm{dn}\,\xi^* + \mathrm{dn}\,\xi) \pm \frac{\mathrm{i}\omega'}{q}(\mathrm{dn}\,\xi^* - \mathrm{dn}\,\xi)\right]F(\xi,\xi^*). \tag{77}$$

From (77) we find that:

$$\partial_4 F(\xi,\xi^*) = \frac{\omega}{2c}(\mathrm{dn}\,\xi^* + \mathrm{dn}\,\xi)F(\xi,\xi^*) \tag{78}$$

$$\partial_5 F(\xi,\xi^*) = \frac{\omega'}{2cq}(\mathrm{dn}\,\xi^* - \mathrm{dn}\,\xi)F(\xi,\xi^*). \tag{79}$$

We now consider the case when the real periods of the functions $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$ are much smaller than their imaginary periods, which means that the modulus k fulfils the condition: $k \ll 1$, which with the variables t and u expressed in the same units implies that the frequencies fulfil the inequality

$$\omega' \ll \omega. \tag{80}$$

This allows the approximative expressions for the elliptic functions $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$ to be applied from [30] which are valid with the accuracy to k^2-terms. These expressions will serve in the calculation of the derivatives in (78) and (79). For brevity, in the following expressions we replace the variables ξ, ξ^* in (71) by $t + \mathrm{i}u$ and $(t - \mathrm{i}u)$, respectively. We now firstly utilize the formulae for the products of Jacobian elliptic functions [30] which appear in the function $F(\xi,\xi^*)$ in (74), and secondly the formulae relating the Jacobian elliptic functions of an imaginary argument, connected with the modulus k, with those of a real argument, connected with the modulus $k' = \sqrt{1-k^2}$. We next apply the approximate expressions for the Jacobian elliptic functions of a real argument, valid for $k \ll 1$, and hence k' close to 1. With the accuracy to k^2-terms we then find the expressions:

$$\mathrm{dn}(t + \mathrm{i}u, k) + \mathrm{dn}(t - \mathrm{i}u, k) = 2 + k^2(\sinh^2 u - 2\sinh^2 u \sin^2 t - \sin^2 t) = A \tag{81}$$

$$\mathrm{dn}(t + \mathrm{i}u, k) - \mathrm{dn}(t - \mathrm{i}u, k) = -\mathrm{i}k^2 \sinh u \sin 2t = B. \tag{82}$$

We now write the solution of (9) or (23) in the form

$$\Psi = \begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} = \begin{bmatrix}\phi_1(\vec{r})\\ \phi_2(\vec{r})\\ \phi_3(\vec{r})\\ \phi_4(\vec{r})\end{bmatrix} F(\xi,\xi^*). \tag{83}$$

Since the function $F(\xi,\xi^*)$ does not vanish in the whole complex plane, we can remove it from (9) or (23) after having performed the differentiation with respect to x_4 and x_5 according to equations (78) and (79).

We now notice that owing to (80) the period of the variable t is very small compared to that of the variable u. In the approximate expressions in (81) and (82) the periodic character of the elliptic functions with respect to the variable u disappeared. It seems to be admissible to average with respect to the t-variable the terms on the right-hand sides of (81) and (82). We then obtain:

$$\langle A\rangle = 2 - \tfrac{1}{2}k^2 \qquad \langle B\rangle = 0. \tag{84}$$

These averages then enter into equations containing the derivatives ∂_4 and ∂_5. We see that if a solution of the five-dimensional Dirac equation is written in the form of (83), then with the accuracy to k^2-terms the derivative with respect to the variable x_5 vanishes, after averaging over the variable t. Consequently, if averaging over the small-period variable t is accepted, then with the accuracy to k^2-terms the influence of the fifth dimension on the energy levels will reduce to the k^2-term in (84).

We notice that in the iterated equation in (25) the term $-2\mathrm{i}\kappa\,\partial_5\Psi$ vanishes with the accuracy to k^2-terms, after averaging over the small-period variable t.

7. Electron in Coulomb and central gravitational fields

In (9) we put $a_\mu = 0$, $\mu = 1, 2, 3$, $a_4 = \mathrm{i}V/\hbar c$, $a_5 = W/\hbar c$, with $W = m\chi$, and with the irrep for the γ_αs in (59), we obtain the set of equations:

$$(\partial_x - \mathrm{i}\partial_y)\psi_4 + \mathrm{i}\partial_z\psi_1 + [(V+W)/\hbar c + \kappa]\psi_3 = -(\partial_4 + \mathrm{i}\partial_5)\psi_3 \tag{85}$$

$$(\partial_x + \mathrm{i}\partial_y)\psi_3 + \mathrm{i}\partial_z\psi_2 - [(V-W)/\hbar c - \kappa]\psi_4 = (\partial_4 - \mathrm{i}\partial_5)\psi_4 \tag{86}$$

$$-(\partial_x - \mathrm{i}\partial_y)\psi_2 - \mathrm{i}\partial_z\psi_3 - [(V-W)/\hbar c - \kappa]\psi_1 = (\partial_4 - \mathrm{i}\partial_5)\psi_1 \tag{87}$$

$$-(\partial_x + \mathrm{i}\partial_y)\psi_1 - \mathrm{i}\partial_z\psi_4 + [(V+W)/\hbar c + \kappa]\psi_2 = -(\partial_4 + \mathrm{i}\partial_5)\psi_2. \tag{88}$$

In (85) through (88) the energy and mass terms are connected with the same spinor component as they are in the set of equations following from the original Dirac equation (58) with the γ_α-irrep in [25, 27, 28]. We assume that a solution of this set of equations has the form in (83). For the respective spinor components ϕ_α, $\alpha = 1, \ldots, 4$ we introduce those of the sum of the spinors $\Phi_1^{(a)} + \Phi_2^{(a)}$ in (66), denoted by $\phi_1^{(a)}$, $\phi_4^{(a)}$ and $\phi_2^{(a)}$, $\phi_3^{(a)}$, respectively. From (85), utilizing the expressions for the operators $\partial_x \pm \mathrm{i}\partial_y$ and ∂_z in spherical coordinates (see, for example, [27, 29]), for the first two terms on the left-hand side we obtain

$$(\partial_x - \mathrm{i}\partial_y)\phi_4^{(a)} = \frac{\mathrm{i}}{2}\frac{\sqrt{j-m_j}}{\sqrt{j+m_j}}\left[\frac{\sqrt{(j-m_j+1)(j-m_j)}}{\sqrt{(j+1)j}}\left(f_j' - \frac{j-\frac{1}{2}}{r}f_j\right)Y_{j+\frac{1}{2}}^{m_j-\frac{1}{2}} - \frac{\sqrt{(j+m_j-1)(j+m_j)}}{\sqrt{j(j-1)}}\left(f_j' + \frac{j+\frac{1}{2}}{r}f_j\right)Y_{j-\frac{3}{2}}^{m_j-\frac{1}{2}}\right] \tag{89}$$

with $f' = \mathrm{d}f/\mathrm{d}r$, and

$$\mathrm{i}\partial_z\phi_1^{(a)} = \frac{\mathrm{i}}{2}\left[\frac{\sqrt{(j-m_j+1)(j+m_j)}}{\sqrt{(j+1)j}}\left(f_j' - \frac{j-\frac{1}{2}}{r}f_j\right)Y_{j+\frac{1}{2}}^{m_j-\frac{1}{2}} + \frac{\sqrt{(j+m_j-1)(j-m_j)}}{\sqrt{j(j-1)}}\left(f_j' + \frac{j+\frac{1}{2}}{r}f_j\right)Y_{j-\frac{3}{2}}^{m_j-\frac{1}{2}}\right] \tag{90}$$

and, consequently

$$(\partial_x - \mathrm{i}\partial_y)\phi_4^{(a)} + \mathrm{i}\partial_z\phi_1^{(a)} = \mathrm{i}\frac{\sqrt{j(j-m_j+1)}}{\sqrt{(j+1)(j+m_j)}}\left(f_j' - \frac{j-\frac{1}{2}}{r}f_j\right)Y_{j+\frac{1}{2}}^{m_j-\frac{1}{2}} \tag{91}$$

since the terms proportional to $Y_{j-3/2}^{m_j-1/2}$ cancel out. The spinor component $\phi_3^{(a)}$ in (66) is proportional to the same spherical harmonic as that appearing in (91). We now divide both sides of (85) by the term $\mathrm{i}\sqrt{j-m_j+1}$, replace the function $f_j(r)$ by $[\sqrt{(j+1)(j+m_j)}/\sqrt{j}]f_j(r)$ and the function $g_j(r)$ by $\sqrt{j+m_j+1}\,g_j(r)$. Consequently, the left-hand side of (85) takes the form

$$f_j' - \frac{j-\frac{1}{2}}{r}f_j - \left(\kappa + \frac{V+W}{\hbar c}\right)g_j. \tag{92}$$

From the left-hand side of (88) we again obtain (92). For the first two terms on the left-hand side of (86) we obtain

$$(\partial_x + \mathrm{i}\partial_y)\phi_3^{(a)} = -\frac{\mathrm{i}}{2}\left[\frac{\sqrt{(j+m_j+2)(j+m_j+1)(j-m_j+1)}}{\sqrt{(j+2)(j+1)}}\left(g_j' - \frac{j+\frac{1}{2}}{r}g_j\right)Y_{j+\frac{3}{2}}^{m_j+\frac{1}{2}}\right.$$
$$\left. - \frac{(j-m_j+1)\sqrt{(j-m_j)}}{\sqrt{(j+1)j}}\left(g_j' + \frac{j+\frac{3}{2}}{r}g_j\right)Y_{j-\frac{1}{2}}^{m_j+\frac{1}{2}}\right] \quad (93)$$

with $g' = \mathrm{d}g/\mathrm{d}r$, and

$$\mathrm{i}\partial_z\phi_2^{(a)} = \frac{\mathrm{i}}{2}\left[\frac{\sqrt{(j+m_j+2)(j+m_j+1)(j-m_j+1)}}{\sqrt{(j+2)(j+1)}}\left(g_j' - \frac{j+\frac{1}{2}}{r}g_j\right)Y_{j+\frac{3}{2}}^{m_j+\frac{1}{2}}\right.$$
$$\left. + \frac{(j+m_j+1)\sqrt{(j-m_j)}}{\sqrt{(j+1)j}}\left(g_j' + \frac{j+\frac{3}{2}}{r}g_j\right)Y_{j-\frac{1}{2}}^{m_j+\frac{1}{2}}\right] \quad (94)$$

and, consequently

$$(\partial_x + \mathrm{i}\partial_y)\phi_3^{(a)} + \mathrm{i}\partial_z\phi_2^{(a)} = \mathrm{i}\frac{\sqrt{(j-m_j)(j+1)}}{\sqrt{j}}\left(g_j' + \frac{j+\frac{3}{2}}{r}g_j\right)Y_{j-\frac{1}{2}}^{m_j+\frac{1}{2}} \quad (95)$$

since the terms proportional to $Y_{j+3/2}^{m_j+1/2}$ cancel out. Since the spinor component $\phi_4^{(a)}$ in (66) is proportional to the same spherical harmonic as that appearing in (95), that spherical harmonic is a common factor in (86) and it can be left out. We now divide both sides of (86) by the term $\mathrm{i}\sqrt{j-m_j+1}$ and next replace the function $f_j(r)$ by $[\sqrt{(j+1)(j+m_j)}/\sqrt{j}]f_j(r)$ and the function $g_j(r)$ by $\sqrt{j+m_j+1}g_j(r)$. The left-hand side of (86) then takes the form

$$g_j' + \frac{j+\frac{3}{2}}{r}g_j - \left(\kappa - \frac{V-W}{\hbar c}\right)f_j. \quad (96)$$

Equation (87) leads to (96). The right-hand sides of (85) and (86) (as well as of (87) and (88)) are determined by (78), (79) and (81) through (84). We obtain the terms $\pm\omega(1-k^2/4)/c$ with the (−) sign for (85) and the (+) sign for (86). Introducing these terms into (85) and (86), respectively, and considering (92) and (96) we obtain the following couple of equations for the radial functions:

$$\frac{\mathrm{d}f_j}{\mathrm{d}r} - \frac{j-\frac{1}{2}}{r}f_j - \left[\kappa - \frac{\omega}{c}\left(1-\frac{1}{4}k^2\right) + \frac{V+W}{\hbar c}\right]g_j = 0 \quad (97)$$

$$\frac{\mathrm{d}g_j}{\mathrm{d}r} + \frac{j+\frac{3}{2}}{r}g_j - \left[\kappa + \frac{\omega}{c}\left(1-\frac{1}{4}k^2\right) - \frac{V-W}{\hbar c}\right]f_j = 0. \quad (98)$$

With:

$$V = -\frac{Ze^2}{r} \qquad \beta = \frac{Ze^2}{\hbar c} \qquad \frac{V}{\hbar c} = -\frac{\beta}{r} \quad (99)$$

and

$$W = -\frac{\Gamma Mm}{r} \qquad \delta = \frac{\Gamma Mm}{\hbar c} \qquad \frac{W}{\hbar c} = -\frac{\delta}{r} \quad (100)$$

where Γ denotes the gravitational constant and M is proton rest mass, introducing the parameters a and μ by the equalities

$$\kappa + \frac{\omega}{c}\left(1-\frac{1}{4}k^2\right) = \frac{mc^2+E}{\hbar c} = \frac{1}{\mu a} \quad (101)$$

$$\kappa - \frac{\omega}{c}\left(1-\frac{1}{4}k^2\right) = \frac{mc^2-E}{\hbar c} = \frac{\mu}{a} \quad (102)$$

where

$$E = \hbar\omega\left(1 - \frac{1}{4}k^2\right) \tag{103}$$

we rewrite (97) and (98) in the form:

$$f_j' - \frac{j - \frac{1}{2}}{r} f_j - \left(\frac{\mu}{a} - \frac{\beta + \delta}{r}\right) g_j = 0 \tag{104}$$

$$g_j' + \frac{j + \frac{3}{2}}{r} g_j - \left(\frac{1}{\mu a} + \frac{\beta - \delta}{r}\right) f_j = 0. \tag{105}$$

From this point on, these equations are integrated in the customary way (see, e.g. [27, 29]). For $r \to \infty$, (104) and (105) have the solutions: $g(r) = C\exp(-r/a)$ and $f(r) = -C\mu\exp(-r/a)$, with C = constant, while for $r \to 0$ we have: $g(r) = Ar^{s-1}$, $f(r) = Br^{s-1}$ with A, B and C being new constants, independent of those in section 5. Inserting the latter solutions into (104) and (105) we find that $s = [(j+1/2)^2 - (\beta^2 - \delta^2)]^{1/2}$. Writing:

$$g(r) = Cr^{s-1}\exp(-r/a)G(r) \tag{106}$$

$$f(r) = -C\mu r^{s-1}\exp(-r/a)F(r) \tag{107}$$

and inserting these functions into (104) and (105) we obtain the equations:

$$\frac{\mathrm{d}G}{\mathrm{d}r} + \left(\frac{s + (j + \frac{1}{2})}{r} - \frac{1}{a}\right)G + \left(\frac{1}{a} + \frac{(\beta - \delta)\mu}{r}\right)F = 0 \tag{108}$$

$$\frac{\mathrm{d}F}{\mathrm{d}r} + \left(\frac{s - (j + \frac{1}{2})}{r} - \frac{1}{a}\right)F + \left(\frac{1}{a} - \frac{\beta + \delta}{\mu r}\right)G = 0. \tag{109}$$

Writing:

$$G(r) + F(r) = v(r) \qquad G(r) - F(r) = w(r) \tag{110}$$

we obtain from (108) and (109) the equations:

$$v' + \frac{s + p_-}{r} v = -\left[\left(j + \frac{1}{2}\right) - p_+\right]\frac{w}{r} \tag{111}$$

with $v' = \mathrm{d}v/\mathrm{d}r$, and

$$w' + \frac{s - p_-}{r} w - \frac{2}{a} w = -\left[\left(j + \frac{1}{2}\right) + p_+\right]\frac{v}{r} \tag{112}$$

with $w' = \mathrm{d}w/\mathrm{d}r$, where

$$p_\pm = \frac{\mu(\beta - \delta)}{2} \pm \frac{\beta + \delta}{2\mu}. \tag{113}$$

From equations (111) and (112) we obtain the equation

$$rv'' + \left[(2s + 1) - \frac{2r}{a}\right]v' - \frac{2}{a}(s + p_-)v = 0 \tag{114}$$

which is of the form of (202.12) on p 197 of [29] and which together with the respective equation for the function w leads to the condition

$$s + p_- = -n_r \qquad n_r = 0, 1, 2, \ldots \tag{115}$$

and, consequently, to the energy-level formula

$$E = \frac{mc^2[\pm(s + n_r)\sqrt{(s + n_r)^2 + \beta^2 - \delta^2} - \beta\delta]}{(s + n_r)^2 + \beta^2} \tag{116}$$

with s determined before (106), which for $\delta = 0$ and the (+) sign before the first term, and the modulus $k = 0$ in (103), turns into the customary energy-level formula. The energy in (116) is identical with that which follows from (4.22) of [31], when the Bohm–Aharonov and magnetic monopole potentials are left out in the latter.

The second solution of (85) and (86) connected with $\Phi_1^{(b)} + \Phi_2^{(b)}$ in (67) leads to the respective following equations:

$$f_j' + \frac{j + \frac{3}{2}}{r} f_j + \left(\frac{1}{\mu a} - \frac{\beta - \delta}{r}\right) g_j = 0 \tag{117}$$

and

$$g_j' - \frac{j - \frac{1}{2}}{r} g_j + \left(\frac{\mu}{a} + \frac{\beta + \delta}{r}\right) f_j = 0. \tag{118}$$

Exchanging in these equations the functions f_j and g_j and replacing μ by $-\mu$, we obtain from (117) and (118), (105) and (104), respectively, thus determining the second solution which belongs to the same energy level, determined by (116). We observe that the five-dimensional form of the Klein–Gordon equation in (26) can be solved for the central-field case with the help of the function $F(\xi, \xi^*)$ in (74).

8. The Pauli equation

This equation will be derived from (9) in the manner of [23] without introducing a matrix irrep for the γs. We replace the spinor Ψ in (9) by the function u in (16) which we factorize in the form

$$u = wF(\xi, \xi^*) \tag{119}$$

with the function w independent of x_4 and x_5 and apart from that of the form of u in (16), and $F(\xi, \xi^*)$ in (74). With the help of (78), (79) and (81) through (84) we calculate the derivatives with respect to x_4 and x_5, and with E in (104), and $mc^2 = E_0$ we obtain the equation

$$\left[\sum_{k=1}^{3} \gamma_k(\partial_k - \mathrm{i}a_k) - \gamma_4 \frac{E - V}{\hbar c} - \mathrm{i}\gamma_5 \frac{E_0 + W}{\hbar c}\right] w = 0. \tag{120}$$

We write the function w in (119) in the form:

$$w = (1 + \mathrm{i}\gamma_4\gamma_5)w^+ + (1 - \mathrm{i}\gamma_4\gamma_5)w^- \tag{121}$$

and notice that:

$$(\gamma_4 \pm \mathrm{i}\gamma_5)(\gamma_4 \mp \mathrm{i}\gamma_5) = 2(1 \mp \mathrm{i}\gamma_4\gamma_5) \tag{122}$$

$$(\gamma_4 \pm \mathrm{i}\gamma_5)(\gamma_4 \pm \mathrm{i}\gamma_5) = 0. \tag{123}$$

Introducing into (120) the function w in (121), multiplying the resulting equation from the left side firstly by $(\gamma_5 - \mathrm{i}\gamma_4)$ and secondly by $(\gamma_5 + \mathrm{i}\gamma_4)$, we obtain in the first case the equation

$$\sum_{k=1}^{3} \mathrm{i}\gamma_5\gamma_k(\partial_k - \mathrm{i}a_k)w^+ - \frac{E - V - E_0 - W}{\hbar c} w^- = 0 \tag{124}$$

and in the second case the equation:

$$\sum_{k=1}^{3} \mathrm{i}\gamma_5\gamma_k(\partial_k - \mathrm{i}a_k)w^- + \frac{E - V + E_0 + W}{\hbar c} w^+ = 0. \tag{125}$$

To eliminate the function w^+ from this pair of equations we apply to (125) from the left the operator

$$Q\frac{\hbar c}{E-V+E_0-W}=\sum_{k=1}^{3}\mathrm{i}\gamma_5\gamma_k(\partial_k-\mathrm{i}a_k)\frac{\hbar c}{E-V+E_0-W} \tag{126}$$

and obtain the equation

$$Q\frac{\hbar c}{E-V+E_0-W}Qw^-+Q\frac{E-V+E_0+W}{E-V+E_0-W}w^+=0 \tag{127}$$

assuming that $W\ll E_0=mc^2$, and taking into account (124), we obtain the equation for the function w^-

$$Q\left[\frac{\hbar c}{E-V+E_0-W}Qw^-\right]+\frac{E-V-E_0-W}{\hbar c}w^-=0. \tag{128}$$

The first term in (128) splits into two terms:

$$\mathrm{I}=\frac{\hbar c}{E-V+E_0-W}Q^2w^- \tag{129}$$

and

$$\mathrm{II}=\frac{\hbar c}{(E-V+E_0-W)^2}\sum_{k=1}^{3}\mathrm{i}\gamma_5\gamma_k\left(\frac{\partial V}{\partial x_k}+\frac{\partial W}{\partial x_k}\right)Qw^-. \tag{130}$$

The first term is equal to

$$\mathrm{I}=\frac{\hbar c}{E-V+E_0-W}\left[\sum_{k=1}^{3}(\partial_k-\mathrm{i}a_k)^2+\frac{e}{\hbar c}\vec{\sigma}\cdot\vec{B}\right]w^- \tag{131}$$

where $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ in the Cartesian coordinates, as after (59), and $\vec{B}=\operatorname{curl}\vec{A}$, with $\vec{A}$ denoting the vector potential. The second term is equal to

$$\begin{aligned}\mathrm{II}=&\frac{ce}{(E-V+E_0-W)^2}[\vec{\sigma}\cdot(\vec{E}\times\vec{p})-\mathrm{i}(\vec{E}\cdot\vec{p})]w^-\\&+\frac{c}{(E-V+E_0-W)^2}[\vec{\sigma}\cdot(\vec{F}\times\vec{p})-\mathrm{i}(\vec{F}\cdot\vec{p})]w^-\end{aligned} \tag{132}$$

with $\partial V/\partial x_k=-eE_k$, and $\vec{p}=(p_1,p_2,p_3)$, in the Cartesian coordinates, with $p_k=-\mathrm{i}\hbar(\partial_k-\mathrm{i}a_k)$, $k=1,2,3$ and $\partial W/\partial x_k=-F_k$. Adding (131) and (132), and multiplying the resulting equation by $\hbar(E-V+E_0-W)/(2mc)$, we obtain the equation

$$\begin{aligned}&\left[\frac{\hbar^2}{2m}\sum_{k=1}^{3}(\partial_k-\mathrm{i}a_k)^2+\frac{(E-V-W)^2-E_0^2}{2E_0}\right]w^-\\&\quad=-\left[\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}+\frac{\hbar\vec{\sigma}\cdot[(e\vec{E}+\vec{F})\times\vec{p}]-\mathrm{i}\hbar(e\vec{E}+\vec{F})\cdot\vec{p}}{2m(E-V+E_0-W)}\right]w^-.\end{aligned} \tag{133}$$

Omitting in this equation W and $\vec{F}$ we obtain from it (20) on p 243 of [23], from which the Pauli equation [32, 33] is derived. With $W\ll mc^2$, $v\ll c$ and $E+E_0-V-W\cong 2mc^2$ we have

$$\frac{(E-V-W)^2-E_0^2}{2E_0}\cong E-V-W-mc^2+\frac{m^4v^4}{8m^3c^2} \tag{134}$$

and we obtain from (133) the Pauli equation containing electromagnetic and non-electromagnetic fields in the form

$$\left[\frac{\vec{p}^2}{2m} + E_0 + V + W - \frac{m^4 v^4}{8m^3 c^2} - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}\right] w^- + \frac{\hbar}{4m^2c^2}\{\vec{\sigma}\cdot[\mathrm{grad}(V+W)\times\vec{p}] - \mathrm{i}\,\mathrm{grad}(V+W)\cdot\vec{p}\} w^- = E w^-. \tag{135}$$

Since in (133) and (135) there only appear the products $-\mathrm{i}\gamma_i\gamma_j$ which generate the quaternion group, we can replace the respective four-dimensional matrices representing σ_k, by the two-dimensional Pauli matrices, and in the matrix form of (135) replace the four-component spinor form of the function w^- by a two-component spinor.

To show that w^- is the larger function from the two functions appearing in (121) we write $\vec{p}/m' = \vec{v}$, with m' denoting the velocity-dependent mass, and $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, and then (124) and (125) acquire the respective forms

$$\gamma_5\vec{\gamma}\cdot\vec{v}w^+ + \frac{E - V - E_0 - W}{m'c} w^- = 0 \tag{136}$$

and

$$\gamma_5\vec{\gamma}\cdot\vec{v}w^- - \frac{E - V + E_0 + W}{m'c} w^+ = 0. \tag{137}$$

Writing: $E - E_0 = \Delta E$, $E + E_0 - V + W \cong 2m'c^2$, $\Delta E - V - W \cong m\vec{v}^2/2$ we find from (136) or (137) that $w^+ \sim (v/c)w^-$.

We observe that from the iterated equation for w^- which follows from (124) and (125) and from the expression for the spinor amplitude C^- in (68) specialized for the case of a zero-mass particle, follows the two-component neutrino spinor.

9. Conclusions

A five-dimensional form of the Dirac equation has been discussed. It is related with the alternative form of the original Dirac equation [1]. The five-dimensional equation is covariant with respect to proper, orthochronous rotations in (4 + 1)-dimensional pseudo-orthogonal space. Together with the electromagnetic potential it also contains a scalar non-electromagnetic potential as the fifth component of the potential 5-vector. The requirement of invariance of this five-dimensional equation under the combined operations of charge conjugation, space inversion and time reversal leads to the conclusion that the real fifth coordinate has to change its sign under charge conjugation and simultaneously either under time reversal or under space inversion. Since we are dealing with flat space the latter possibility does not seem to lead to experimentally acceptable consequences. This means that although in the metric expression the fifth coordinate appears with the same sign as the three spatial coordinates, it has the property of a time variable, differing from the ordinary time variable in also being odd under charge conjugation. The odd character of the fifth coordinate under time reversal allows for a new interpretation of its role in the five-dimensional equation. The $\exp(\mathrm{i}\omega t)$ function determining the time-dependence of a stationary state in Minkowski space is replaced by a function constructed from doubly-periodic Jacobian elliptic functions of the complex-time arguments $(\omega t \pm \mathrm{i}\omega' u)/2$ with real ω and ω'. It is shown that in the limit of a vanishing modulus k of the elliptic functions, the correction to the energy levels of an electron in Coulomb and central gravitational fields stemming from the fifth dimension, is proportional to k^2. The integration of the equation containing central fields is conditioned by the application of new four-component eigenspinors of the total angular momentum operator, with non-zero entries in the first and fourth or in the second and third row of the column matrix. These spinors have to be used also for the integration of the alternative form of the Dirac equation in (1) containing Coulomb field. This is due to the presence of γ_5 in that equation. The iterated five-dimensional equation contains the antisymmetric second-rank tensor, defining the electromagnetic and the non-electromagnetic field components. It can be shown that utilizing this tensor, Maxwell–Nordström [3, 21] equations can be rederived. They then are in covariant form with respect to five-dimensional rotations. It can also be shown that the odd character of the real fifth coordinate under time reversal is implied by the Maxwell–Nordström equations, if their invariance under time reversal is required. The Klein–Gordon equation, which is contained in the iterated equation, can be integrated for central fields with the help of the same function constructed from Jacobian elliptic functions as for the Dirac equation. The iterated equation contains among other terms a new term determining the operator of the potential energy of electron spin density in a non-electromagnetic field. The Pauli equation derived from the five-dimensional equation contains a spin–orbit-type coupling term, depending on the non-electromagnetic potential.

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